

Equation of motion method for Full Counting Statistics: Steady state superradiance

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For the multi-mode Dicke model in a transport setting that exhibits collective boson transmissions, we construct the equation of motion for the cumulant-generating function. Approximating the exact system of equations at the level of cumulant-generating function and system operators at lowest order, allows us to recover master equation results of the Full Counting Statistics for certain parameter regimes at very low cost of computation. The thermodynamic limit, that is not accessible with the master equation approach, can be derived analytically for different approximations.

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I. INTRODUCTION

The fluctuations of particles tunneling randomly through a quantum structure are sensitive indicators for particle interactions [1], coherences [2] or collective effects [3]. Originating from the field of quantum optics [4], the theory of Full Counting Statistics (FCS) has proven to be a versatile tool for the evaluation of these fluctuations, that can be conveniently described by the cumulants of the underlying stochastic process [5]. FCS is successfully applied in electronic systems [6, 7], where the charge fluctuations can be measured by a nearby quantum point contact [8]. While for various small-sized quantum systems the FCS can be derived analytically [9], the calculation of the full statistics, i.e., all higher cumulants, of the emitted particles for an arbitrarily large system size is cumbersome due to the required diagonalization of the large Liouvillian $\mathcal{L}(\chi)$, that describes the system dynamics.

Apart from numerical methods the most obvious approach to this problem is to make further approximations concerning the factorization of higher order correlation functions, as is routinely done in, e.g., the Hartree-Fock approach in solid state physics [10]. While this factorization is usually applied at the level of specific observables, e.g., for two-electron processes, in this work we propose a factorization for the correlations of the cumulant-generating function (CGF) and system operators derived within a weak-coupling theory. One of the most basic and yet size-scalable models is the Dicke model [11] for superradiant decay of an initially excited atomic cloud interacting with a radiation field [12]. In the limit of small sample size, the dynamics of the system can be analyzed in the symmetrical angular momentum basis [13] and analytic results for higher correlations of the transient pulse of emitted bosons can be obtained [14]. A semi-classical propagator for transient Dicke superradiance was derived in [15].

In this work, we consider an extended multi-mode Dicke model in a transport setup [16], in a limit where back-action between counted particles and the system state can be neglected. Contrary to previous work we have therefore a steady-state transport setting that al-

lows us to obtain analytic long-time results. For this aim, we construct an equation of motion (EoM) for the CGF, the solution of which requires us to make factorization assumptions for correlations of counting and system operators. We show that in certain limits even factorization at lowest level allows one to retrieve the FCS for arbitrary system size and access the thermodynamic limit. The semi-classical results for transient effects can be obtained as limiting cases [17].

With the recent progress in single-photon detectors [18] and the rising interest in collective boson transport, e.g., in thermal transport [19] or new types of lasers, like phonon and superradiant lasers [20, 21], we expect our findings to be relevant for a broader community.

This work is structured as follows. In Sec. II we introduce the model under consideration and briefly introduce the underlying theory. In Sec. III we present exact EoM for the CGF and correlations. In Sec. IV we consider arbitrary system size N and compare the solutions for different approximations that close the EoM (Sec. IV A). We then derive analytic expressions for the thermodynamic limit of the FCS (Sec. IV B), examine the scaling behavior of the cumulants (Sec. IV C), and discuss the quality of the different approximations (Sec. IV D). We finish in Sec. V with a conclusion.

II. MODEL

Our model consists of two bosonic reservoirs, source S and drain D , that are coupled collectively by a medium of N two-level systems with identical level splitting Ω . It is described by an extended multi-mode Dicke Hamiltonian [16]

$$H = \frac{\Omega}{2} J_z + \sum_{\alpha=S,D} \sum_k \omega_{k\alpha} b_{k\alpha}^\dagger b_{k\alpha} + J_x \sum_{\alpha=S,D} \sum_k \left(h_{k\alpha} b_{k\alpha}^\dagger + h_{k\alpha}^* b_{k\alpha} \right) \quad (1)$$

with collective spin operators $J_{x,z} \equiv \sum_{i=1}^N \sigma_i^{x,z}$, where the operator $b_{k\alpha}^\dagger$ creates a boson of frequency $\omega_{k\alpha}$ in reservoir α with the corresponding coupling constant $h_{k\alpha}$.

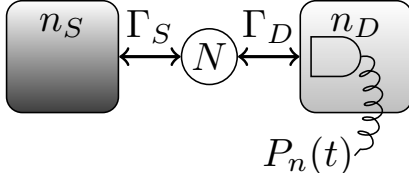


FIG. 1. Sketch of the model: Bosonic excitations are absorbed/emitted with rates $\Gamma_{S/D}$ in and out of the medium of N two-level systems to source or drain, respectively (occupations $n_S > n_D$). The number of total boson emissions into the drain is counted to construct the distribution $P_n(t)$, denoting the probability for n particles effectively emitted into the drain after time t .

We obtain the counting statistics of the total number n of bosons exchanged between the medium and the drain by formally introducing [22] a bookkeeping operator $d^\dagger = \sum_{n=-\infty}^{\infty} |n+1\rangle \langle n|$ in the Hamiltonian via $b_{kD}^\dagger \rightarrow b_{kD}^\dagger \otimes d^\dagger$ and similarly for the annihilation operator. The bookkeeping operator d^\dagger increases the occupation of a virtual detector by one unit for every boson created in the drain, independent on the mode k . In the following, we consider the N -two-level medium and the bookkeeping operator as the system and assume a weak coupling between system and reservoirs. This allows us to derive a Lindblad master equation in Born, Markov, and secular approximation, where we assume initial decoupling of system and bath, a memory-less bath and neglect fast rotating terms in comparison with the system time-scale [23].

With the extension of the Hamiltonian by the bookkeeping operator, the system state $\rho^{(n)} \equiv \langle n | \rho | n \rangle$ is conditioned on the number of bosons n measured in the detector. The master equation assumes the form

$$\dot{\rho}^{(n)}(t) = \mathcal{L}_0 \rho^{(n)} + \mathcal{L}_+ \rho^{(n-1)} + \mathcal{L}_- \rho^{(n+1)}, \quad (2)$$

where \mathcal{L}_\pm describe emissions out of (+) and into (−) the system to and from the drain, while \mathcal{L}_0 describes the remaining evolution. Explicitly, the super-operators are given by [16]

$$\begin{aligned} \mathcal{L}_0 \rho &= -\frac{i\Omega}{2} [J_z, \rho] - \sum_{\alpha=S,D} \frac{\Gamma_\alpha n_\alpha}{2} \{J_- J_+, \rho\} \\ &\quad - \sum_{\alpha=S,D} \frac{\Gamma_\alpha (1+n_\alpha)}{2} \{J_+ J_-, \rho\} \\ &\quad + \Gamma_S [1+n_S] J_- \rho J_+ + \Gamma_S n_S J_+ \rho J_-, \\ \mathcal{L}_+ \rho &= \Gamma_D [1+n_D] J_- \rho J_+, \\ \mathcal{L}_- \rho &= \Gamma_D n_D J_+ \rho J_-, \end{aligned} \quad (3)$$

where $\Gamma_\alpha \equiv 2\pi \sum_k |h_{k\alpha}|^2 \delta(\Omega - \omega_{k\alpha})$ denotes the spontaneous boson emission rate of a single two-level system into a vacuum reservoir α and $\{\cdot, \cdot\}$ is the anticommutator. The numbers n_α denote the stationary occupation of bath modes at system transition frequency Ω : For the case of thermal baths one has, e.g., $n_\alpha = (e^{\beta_\alpha \Omega} - 1)^{-1}$, where β_α is the inverse temperature of bath α .

The probability of counting n bosons after time t is given by $P_n(t) = \text{Tr} \{ \rho^{(n)}(t) \}$. The n -resolved master equation (2) can be Fourier-transformed by introducing a counting field χ via $\rho(\chi, t) \equiv \sum_n \rho^{(n)}(t) e^{in\chi}$ leading to a generalized master equation for collective spontaneous emission and absorption $\frac{d}{dt} \rho(\chi, t) = \mathcal{L}(\chi) \rho(\chi, t)$ with $\mathcal{L}(\chi) = \mathcal{L}_0 + e^{+i\chi} \mathcal{L}_+ + e^{-i\chi} \mathcal{L}_-$. In case of a single reservoir and neglect of the counting field, this reproduces previous master equations [25].

The FCS of the bosonic detection probability $P_n(t)$ for n bosons after time t counted at the drain reservoir is given by the cumulants $\langle\langle n_k(t) \rangle\rangle \equiv (-i\partial_\chi)^k \mathcal{C}(\chi, t)|_{\chi=0}$, denoted by double brackets, where

$$\mathcal{C}(\chi, t) \equiv \ln \text{Tr} \{ \rho(\chi, t) \} = \ln \text{Tr} \{ e^{\mathcal{L}(\chi)t} \rho_0 \} \quad (4)$$

is the CGF, cf. e.g., [26]. In the stationary limit the CGF can be approximated by the eigenvalue $\lambda_0(\chi)$ with smallest modulus, fulfilling $\lambda_0(0) = 0$, such that $\lim_{t \rightarrow \infty} \mathcal{C}(\chi, t) \approx \lambda_0(\chi)t$. Throughout this work we are investigating the long-time current cumulants $\langle\langle I_k \rangle\rangle \equiv \lim_{t \rightarrow \infty} \frac{d}{dt} \langle\langle n_k(t) \rangle\rangle$.

Calculations are conveniently performed in the angular momentum basis, where $j = \frac{N}{2}$ and

$$\begin{aligned} J_z |j, m\rangle &= 2m |j, m\rangle, \\ J_\pm |j, m\rangle &= \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle, \\ \mathbf{J}^2 |j, m\rangle &= 4j(j+1) |j, m\rangle, \end{aligned} \quad (5)$$

with $\mathbf{J} = (J_x, J_y, J_z)^T$, and $[J_+, J_-] = J_z$. In our model the system size N is fixed, so that we can treat the expectation value $\langle \mathbf{J}^2 \rangle = \mathbf{J}^2 = 2N(\frac{N}{2} + 1)$ as a constant. For later use note that $J_- J_+ = \frac{1}{4} [\mathbf{J}^2 - J_z^2] - \frac{1}{2} J_z$ and $[J_z, J_\pm] = \pm 2J_\pm$.

III. EXACT RESULTS

For the stationary properties of the expectation values $\langle J_z^\alpha \rangle$ we have an exact benchmark: Even for the case of non-equilibrated reservoirs the stationary state of the system, defined by $\mathcal{L}(0)\bar{\rho} = 0$, is a thermal one, $\bar{\rho} \propto e^{-\frac{\beta\Omega}{2} J_z}$, with a temperature $\bar{\beta}^{-1}$ that corresponds to coupling the system to a single fictitious reservoir with the weighted average occupation

$$\bar{n} = \frac{\Gamma_S n_S + \Gamma_D n_D}{\Gamma_S + \Gamma_D} \equiv \frac{1}{e^{\bar{\beta}\Omega} - 1}. \quad (6)$$

Such simple average occupations exist also for an arbitrary number of baths whenever the operator structures of the couplings are identical [24]. Since in addition our system only supports a single allowed transition frequency, it effectively thermalizes at some average temperature even when coupled to baths at different temperatures.

Using this effective thermalization we find the ratio of the matrix elements $\bar{\rho}_m = \langle jm | \bar{\rho} | jm \rangle$ of the tri-diagonal

Liouvillian $\frac{\bar{\rho}_{m+1}}{\bar{\rho}_m} = \frac{\bar{n}}{\bar{n}+1}$. In connection with normalization $\sum_m \bar{\rho}_m = 1$, this implies that the steady state expectation values

$$\langle \bar{J}_z^k \rangle_{ex} = \text{Tr} \{ J_z^k \bar{\rho} \}, \quad (7)$$

with $k = 1, 2, \dots$ can be calculated exactly (not shown for brevity) for arbitrary system sizes N .

The evolution of the CGF for the emitted bosons at the drain can be calculated using $\mathcal{C}(\chi, t) = \ln \langle e^{i\chi} \rangle = \ln \sum_n e^{i\chi} P_n(t)$, which yields

$$\begin{aligned} \frac{d}{dt} \mathcal{C}(\chi, t) &= \frac{\sum_n e^{i\chi} \text{Tr} \{ \dot{\rho}^{(n)} \}}{\langle e^{i\chi} \rangle} \\ &= n_D (f[\chi] - f[-\chi]) \frac{\langle e^{i\chi} J_z \rangle}{\langle e^{i\chi} \rangle} \\ &\quad + \frac{n_D}{2} (f[\chi] + f[-\chi]) \left[J^2 - \frac{\langle e^{i\chi} J_z^2 \rangle}{\langle e^{i\chi} \rangle} \right] \\ &\quad + \frac{f[\chi]}{2} \left[J^2 - \frac{\langle e^{i\chi} J_z^2 \rangle}{\langle e^{i\chi} \rangle} + 2 \frac{\langle e^{i\chi} J_z \rangle}{\langle e^{i\chi} \rangle} \right], \end{aligned} \quad (8)$$

where

$$f[\chi] \equiv \frac{1}{2} \Gamma_D (e^{i\chi} - 1) \quad (9)$$

contains the counting field. Thus, the time-evolution of the CGF couples only to two other correlations $\langle e^{i\chi} J_z \rangle$ and $\langle e^{i\chi} J_z^2 \rangle$. We therefore calculate the EoM for correlations of arbitrary order $\alpha \geq 1$, which is given by

$$\begin{aligned} \frac{d}{dt} \langle e^{i\chi} J_z^\alpha \rangle &\equiv \sum_n \text{Tr} \{ e^{i\chi} J_z^\alpha \dot{\rho}^{(n)}(t) \} \\ &= \left\langle e^{i\chi} \left[\Gamma_S n_S [(J_z + 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_- \right. \right. \\ &\quad + \Gamma_S (1 + n_S) [(J_z - 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_+ \\ &\quad + \Gamma_D n_D [e^{-i\chi} (J_z + 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_- \\ &\quad \left. \left. + \Gamma_D (1 + n_D) [e^{i\chi} (J_z - 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_+ \right] \right\rangle, \end{aligned} \quad (10)$$

with

$$\mathbf{X}_\pm = \frac{1}{4} (J^2 - J_z^2 \pm 2J_z), \quad (11)$$

and thus couples the evolution of $\langle e^{i\chi} J_z^\alpha \rangle$ to all powers of $\langle e^{i\chi} J_z^{\alpha'} \rangle$ where $\alpha' \in \{0, 1, \dots, \alpha+2\}$. This leads to a hierarchy problem, such that the straightforward solution of these EoMs without factorizations is difficult for large $j = \frac{N}{2}$. Equations (8) – (11) are the first central result of this paper.

For $N = 1$ (i.e., $j = \frac{1}{2}$) however, the above system can be solved exactly. For this special case we have $J_z = \sigma_z$ and $\sigma_z^2 = \mathbb{1}$ and thus we can solve the system of equations without the need of any factorization. We introduce the

abbreviation $A(\chi, t) \equiv \frac{\langle e^{i\chi} J_z \rangle}{\langle e^{i\chi} \rangle}$ and obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{C}(\chi, t) &= - (e^{-i\chi} - 1) \frac{\Gamma_D}{2} n_D [A(\chi, t) - 1] \\ &\quad + (e^{+i\chi} - 1) \frac{\Gamma_D}{2} [1 + n_D] [A(\chi, t) + 1] \\ \frac{d}{dt} A(\chi, t) &= \underbrace{\frac{d}{dt} \frac{\langle e^{i\chi} J_z \rangle}{\langle e^{i\chi} \rangle}}_I - A(\chi, t) \left(\frac{d}{dt} \mathcal{C}(\chi, t) \right), \end{aligned} \quad (12)$$

with

$$\begin{aligned} I &= -\Gamma_S n_S [A(\chi, t) - 1] - \Gamma_S [1 + n_S] [A(\chi, t) + 1] \\ &\quad - \frac{\Gamma_D}{2} n_D (e^{-i\chi} + 1) [A(\chi, t) - 1] \\ &\quad - \frac{\Gamma_D}{2} [1 + n_D] (e^{+i\chi} + 1) [A(\chi, t) + 1]. \end{aligned} \quad (13)$$

The quadratic equation for $A(\chi, t)$, cf. Eq. (12), can be solved and inserted in the equation for the CGF. Since we are interested in long-time dynamics, we can directly take derivatives with respect to χ and send $t \rightarrow \infty$ to obtain the corresponding stationary cumulants. Note that to circumvent the quadratic equation one could alternatively calculate the dynamics of the moment-generating function $\mathcal{M}(\chi, t) \equiv \langle e^{i\chi} \rangle$, where one has to solve a system of coupled linear equations and then construct the cumulants from the moments. As expected, the thus obtained CGF coincides in the long-time limit with the eigenvalue $\lambda_0(\chi)$ with smallest modulus, obtained by diagonalizing the corresponding 2×2 Liouvillian, and is given by

$$\begin{aligned} \mathcal{C}^{N=1}(\chi, t) &= -\frac{1}{2} t \left[\Gamma_S (1 + 2n_S) + \Gamma_D (1 + 2n_D) \right. \\ &\quad - \left\{ \Gamma_S \Gamma_D [e^{-i\chi} (1 + n_S) n_D + e^{+i\chi} n_S (1 + n_D)] \right. \\ &\quad \left. \left. + \frac{1}{4} [2\Gamma_S \Gamma_D + (\Gamma_S (1 + 2n_S))^2 + (\Gamma_D (1 + 2n_D))] \right\}^{\frac{1}{2}} \right]. \end{aligned} \quad (14)$$

In contrast, for large N it is much more difficult to obtain the CGF, as already the size of the Liouvillian grows as $(N+1) \times (N+1)$. However, for the long-time limit of the first cumulant $\langle \langle I_1 \rangle \rangle$ we can make use of Eq. (7) and the fact that trace conservation implies $\text{Tr} \{ \mathcal{L}(0) \rho \} = 0$, such that we find

$$\begin{aligned} \langle \langle I_1^{\text{ME}} \rangle \rangle &= -i \text{Tr}_S \{ \mathcal{L}'(0) \bar{\rho} \} \\ &= (n_S - n_D) \frac{\Gamma_S \Gamma_D}{\Gamma_S + \Gamma_D} \sigma_N, \\ \sigma_N &= \frac{(N - 2\bar{n}) (1 + \bar{n})^{N+1} + \bar{n}^{N+1} (2 + N + 2\bar{n})}{(1 + \bar{n})^{N+1} - \bar{n}^{N+1}}, \end{aligned} \quad (15)$$

where the superscript ME denotes the master equation solution. This result (which we obtained previously [16]) gives us one analytic benchmark for approximate solutions.

Due to $[\mathcal{L}'(\chi), \mathcal{L}(\chi)] \neq 0$, the above approach yields no simple analytic results for higher cumulants (that require higher derivatives with respect to χ). The corresponding expressions still involve the full χ -dependent Liouvillian, that needs to be diagonalized for a solution. To bypass the expensive numerical calculation, which becomes unfeasible already for moderate N , we have to apply additional approximations.

IV. APPROXIMATE EQUATIONS OF MOTION

Eq. (8) and (10) constitute the EoM for the full CGF. To circumvent the problem of the hierarchy of operator equations, we introduce a lowest-order factorization at the level of the CGF and system operators, and show that already this very crude assumption gives useful results comparable to the master equation solution.

The simplest way to make Eq. (8) solvable is to add one further assumption to the Born-Markov-Secular approximation, namely that the counting operator n has no correlations in any order α with the system operators

$$\langle e^{in\chi} J_z^\alpha \rangle \approx \langle e^{in\chi} \rangle \langle J_z^\alpha \rangle. \quad (16)$$

This assumption is strictly valid only when the statistics of particles counted at the drain and the system state are independent. Inserting the approximation Eq. (16) into Eq. (8) yields another central result of this work: an explicit equation for the CGF

$$\begin{aligned} \frac{d}{dt} \mathcal{C}(\chi, t) = n_D (f[\chi] - f[-\chi]) \langle J_z \rangle \\ + \frac{n_D}{2} (f[\chi] + f[-\chi]) [\mathbf{J}^2 - \langle J_z^2 \rangle] \\ + \frac{f[\chi]}{2} [\mathbf{J}^2 - \langle J_z^2 \rangle + 2 \langle J_z \rangle], \end{aligned} \quad (17)$$

with $f[\chi]$ given by Eq. (9).

Independent of the choice of solution for $\langle J_z^{(1,2)} \rangle$, taking derivatives with respect to χ shows that all time derivatives of odd ($\langle \langle n_{2k+1} \rangle \rangle$) and even ($\langle \langle n_{2k} \rangle \rangle$) cumulants are identical, respectively, and are given by

$$\begin{aligned} \frac{d}{dt} \frac{\langle \langle n_{2k+1} \rangle \rangle}{\Gamma_D} = n_D \langle J_z \rangle + \frac{1}{4} [\mathbf{J}^2 - \langle J_z^2 \rangle + 2 \langle J_z \rangle], \quad (18) \\ \frac{d}{dt} \frac{\langle \langle n_{2k} \rangle \rangle}{\Gamma_D} = \frac{1}{2} \left[\left(n_D + \frac{1}{2} \right) (\mathbf{J}^2 - \langle J_z^2 \rangle) + 2 \langle J_z \rangle \right]. \end{aligned}$$

Therefore, provided that Eq. (16) is approximately valid, the statistics of the model can be retrieved with knowledge of only $\langle J_z \rangle$ and $\langle J_z^2 \rangle$.

A. Approximate solutions for large spin

For our specific system we are able to obtain exact steady state expressions for the expectation values of the system operator for arbitrary power $\langle J_z^\alpha \rangle$, cf. Eq. (7). If

one applied our approach to other systems, this would generally not be the case. To close the system of equations we would therefore have to calculate the EoM for the system operators as well, even when factoring correlations of the type of Eq. (10).

To elucidate the validity of the central approximation Eq. (16), we also calculate solutions of Eq. (17) using approximate expressions for the evolution of the system operators. For this aim, using Eq. (3) yields an EoM for powers of J_z given by

$$\begin{aligned} \frac{d}{dt} \frac{\langle J_z^\alpha \rangle}{\Gamma} = \bar{n} \langle [(J_z + 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_- \rangle \\ + (\bar{n} + 1) \langle [(J_z - 2 \cdot \mathbb{1})^\alpha - J_z^\alpha] \mathbf{X}_+ \rangle, \end{aligned} \quad (19)$$

where $\Gamma = \Gamma_S + \Gamma_D$, \mathbf{X}_\pm are given in Eq. (11) and \bar{n} is given in Eq. (6). Since $\mathbf{X}_\pm \propto J_z^2$, the evolution of an operator J_z^α couples to all operators $J_z^{\alpha'}$ with $\alpha' \in \{1, 2, \dots, \alpha + 1\}$. The system of coupled linear differential equations for $\langle J_z^\alpha \rangle$ gives again rise to a hierarchy problem, now on the level of system operators, and can be solved approximately for large j . Factoring the expectation value at any level (e.g. $\langle J_z^{\alpha+\alpha'} \rangle \approx \langle J_z^\alpha \rangle \langle J_z^{\alpha'} \rangle$) will lead to a coupled set of non-linear differential equation, that in the time-dependent case requires numerical solutions. In our approach we are interested in steady-state results and will therefore always use long-time expectation values.

Approximation 1

To estimate the errors introduced by the factorizations on top of the general one (16), we can use solutions of Eq. (17) as a benchmark, where the exact stationary results $\langle J_z^1 \rangle_{ex}$ and $\langle J_z^2 \rangle_{ex}$ of Eq. (7) for $\alpha = 1, 2$ are inserted. Since we have applied our general factorization Eq. (16) even when using the exact expressions we will denote this solution as Approximation 1. In the following we differentiate between two further levels of factorization. Higher order approximations can easily be constructed.

Approximation 2

Here, in addition to (16) we take only the dynamics of $\langle J_z \rangle$ into account, assume $\langle J_z^2 \rangle \approx \langle J_z \rangle^2$ in Eq. (19) with $\alpha = 1$ (Approximation 2). This is valid whenever the variance vanishes and leads to a single quadratic equation,

$$\frac{d}{dt} \frac{\langle J_z \rangle}{\Gamma} = -(2\bar{n} + 1) \langle J_z \rangle + \frac{1}{2} \langle J_z \rangle^2 - \frac{1}{2} \mathbf{J}^2, \quad (20)$$

which can be solved analytically for all times. Since we are interested in steady state dynamics of the FCS, we take the long-time limit and insert the solution in Eq. (17) where we again assume $\langle J_z^2 \rangle \approx \langle J_z \rangle^2$.

Approximation 3

Alternatively, again in addition to (16) we take $\langle J_z^2 \rangle$ in Eq. (19) for $\alpha = 1$ into account and factorize third order correlations $\langle J_z^3 \rangle \approx \langle J_z \rangle \langle J_z^2 \rangle$ for Eq. (19) with $\alpha = 2$ (Approximation 3). This yields a coupled system of (non-linear) differential equations,

$$\begin{aligned} \frac{d}{dt} \frac{\langle J_z \rangle}{\Gamma} &= -(2\bar{n} + 1) \langle J_z \rangle + \frac{1}{2} \langle J_z^2 \rangle - \frac{1}{2} \mathbf{J}^2, \\ \frac{d}{dt} \frac{\langle J_z^2 \rangle}{\Gamma} &= (2\bar{n} + 1) [\mathbf{J}^2 - 3 \langle J_z^2 \rangle] - [\mathbf{J}^2 - 2] \langle J_z \rangle \\ &\quad + \langle J_z^2 \rangle \langle J_z \rangle, \end{aligned} \quad (21)$$

that in general has to be solved numerically. However, the steady state solution can be obtained analytically and again is inserted into Eq. (17), where apart from Eq. (16) we now do not have to assume further factorizations.

B. Thermodynamic limit for the cumulant-generating function

The full CGF for the master equation solution is not accessible for arbitrary system size, since the diagonalization of the large Liouvillian is computationally cumbersome. For the approximate case however, solutions can be easily derived by solving Eq. (17) with the different approximations and we are thus able to examine the thermodynamic limit of infinite system size.

The obtained expressions for the approximate CGFs, with α marking the applied approximation, are $\mathcal{C}^\alpha(\chi, t) = \mathbf{F}(\chi, t) \tilde{\mathcal{C}}^\alpha$ with

$$\begin{aligned} \tilde{\mathcal{C}}^1 &= \frac{(\bar{n} + 1)(N - 2\bar{n}) + \bar{n} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^N (N + 2\bar{n} + 2)}{\bar{n} \left[1 - \left(\frac{\bar{n}}{1 + \bar{n}} \right)^N \right] + 1}, \\ \tilde{\mathcal{C}}^2 &= -(2\bar{n} + 1) + \sqrt{(2\bar{n} + 1)^2 + N(2 + N)}, \\ \tilde{\mathcal{C}}^3 &= -\frac{1}{2} (3 + 6\bar{n} - (2\bar{n} + 1)^{-1}) \\ &\quad + \sqrt{\frac{1}{4} (3 + 6\bar{n} - (2\bar{n} + 1)^{-1})^2 + N(2 + N)}, \end{aligned} \quad (22)$$

where the common function

$$\begin{aligned} \mathbf{F}(\chi, t) &= \Gamma_D t \left[(e^{i\chi} - 1)(n_D + 1)\bar{n} \right. \\ &\quad \left. + (e^{-i\chi} - 1)n_D(\bar{n} + 1) \right], \end{aligned} \quad (23)$$

is shared by all approximations. The form of Eq. (23) shows that the transport process is a balance of drain occupation and effective average occupation, and the choice of approximation yields an overall scaling of the corresponding emission and absorption processes.

These expressions allow us to directly verify the thermodynamic limit for $N \rightarrow \infty$ in specific limits.

For $\bar{n} \ll N$ we have a linear scaling of the CGF in the system size and obtain the same result for all approximations

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{C}^\alpha(\chi, t) = \mathbf{F}(\chi, t). \quad (24)$$

If we perform the thermodynamic limit and simultaneously require $\bar{n} \gg N$, we are always in the super-transmittance regime [16]. Scaling only the source occupation, with a finite drain occupation, we recover the quadratic scaling of the CGF on the system size for all approximations

$$\begin{aligned} \lim_{n_S \rightarrow \infty} \mathcal{C}^\alpha(\chi, t) &= \frac{\Gamma_D}{x_\alpha} N(N + 2) \left[(e^{+i\chi} - 1)(1 + n_D) \right. \\ &\quad \left. + (e^{-i\chi} - 1)n_D \right] t, \end{aligned} \quad (25)$$

where the prefactor gives a deviating scaling for Approximation 2 ($x_2 = 4$) but agrees with the exact solution ($x_1 = x_3 = 6$) for the high thermo-bias, cf. [16], otherwise. This indicates that the coherent effect of "super-transmittance" [16] is correctly found also by the factorization approach.

As a further benchmark for our approximations we check the limit of low thermo-bias, where for low source occupations $n_S \ll 1$ and vanishing drain occupations $n_D = 0$ we again exactly recover previous findings (cf. [16]) for the master equation solution

$$\mathcal{C}(\chi, t) = \frac{\Gamma_S \Gamma_D}{\Gamma_S + \Gamma_D} (e^{+i\chi} - 1) n_S N t, \quad (26)$$

with all approximations.

C. Scaling of stationary cumulants

To quantify the errors which are introduced by the different approximations, we compare the corresponding long-time cumulants to the full master equation solution, which were derived in [16].

First, we compare the ratio of approximate and master equation solution for the steady state of the first current cumulant $\frac{\langle\langle I^{AP} \rangle\rangle}{\langle\langle I^{ME} \rangle\rangle}$. The master equation solution is given in Eq. (15) and the approximate solutions are calculated with Eq. (22).

Interestingly, approximation 1 yields the exact analytic expression for arbitrary source or drain occupations n_α , tunneling rates Γ_α and system size N for the first stationary current cumulant, as shown by the black, solid line in all panels of Fig. 2.

For low source occupation n_S , cf. top panel of Fig. 2, approximations 2 (red, dotted line) and 3 (blue, dashed line) yield results comparable to the exact solution. For higher source occupation n_S (middle and bottom panel), approximation 2 shows a deviation from the exact results for small system size. With higher occupations it is sustained up to larger system sizes. Approximation 3 shows

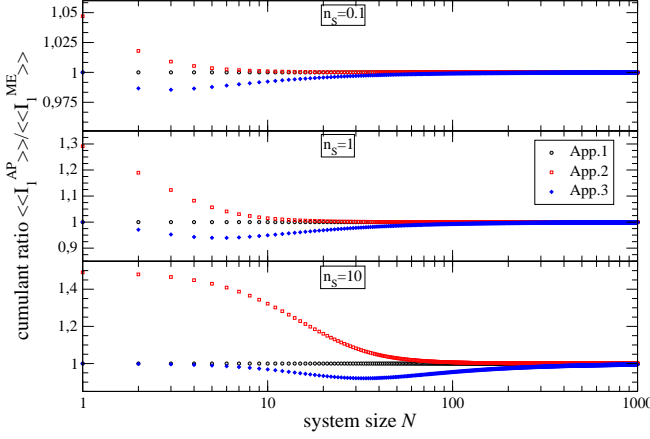


FIG. 2. (Color Online) Logarithmic plot of the ratio of approximated and exact master equation solution for the first steady state cumulant $\frac{\langle\langle I_1^{AP} \rangle\rangle}{\langle\langle I_1^{ME} \rangle\rangle}$ versus the system size N for the different approximations 1 (grey filled, black circles), 2 (red squares), 3 (filled, blue diamonds), cf. Eq. (22), computed for different dimensionless source occupations $n_S = 0.1$ (top), $n_S = 1$ (middle) and $n_S = 10$ (bottom). Note the different scales on the ordinate axes. Other parameters: $n_D = 0$.

a different type of deviation. The position of the maximal deviation is dependent on the source occupation and shifted to larger system size for higher occupations. Additionally, as expected all approximations approach the exact solution with larger system sizes.

The approximate solutions for the second cumulant $\langle\langle I_2^{AP} \rangle\rangle$ can be analytically calculated from Eq. (22). However, for the master equation results $\langle\langle I_2^{ME} \rangle\rangle$, we have to revert to numerical solutions. Since we are interested in the validity of our factorization Eq. (16) for a broader range of occupations, we compare in Fig. 3 the approximated second cumulants $\langle\langle I_2^{AP} \rangle\rangle$ and the numerical master equation solution $\langle\langle I_2^{ME} \rangle\rangle$ for different system sizes $N = \{5, 10, 20, 40, 80\}$ over the range of source occupations $n_S \in \{0.01 \dots 1000\}$ (setting $n_D = 0$).

While for low source occupations all approximations show perfect agreement with the master equation solution (thick, black dot-dashed line), approximation 2 (red, dotted line) shows a constant deviation to the master equation solution for large source occupations. However, approximations 1 (thin, black solid line) and 3 (blue, dashed line) show excellent agreement, again. This is valid for all considered system sizes. For intermediate source occupations all approximations show large deviations from the master equation solution in higher cumulants.

D. Quality of approximations

The quality of the approximations can be understood by evaluating the steady state expression $\langle J_z^\alpha \rangle$, cf.

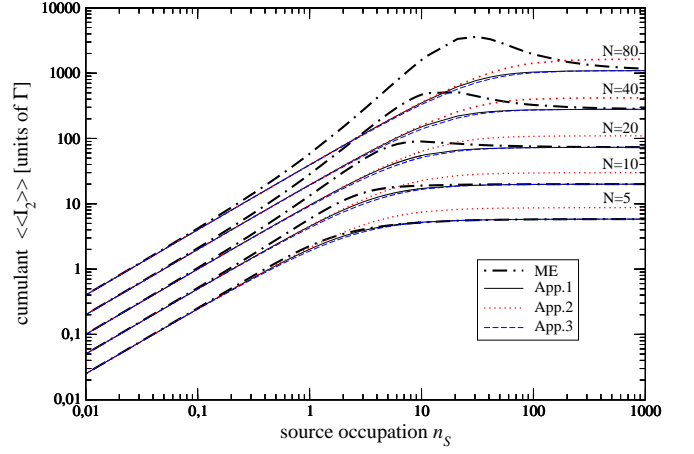


FIG. 3. (Color Online) Double logarithmic plot of the second steady state cumulant $\langle\langle I_2 \rangle\rangle$ in units of $\Gamma = \Gamma_S = \Gamma_D$ versus the dimensionless source occupation n_S for the numerical master equation solution (bold dot-dashed, black) and the different approximations 1 (solid, black), 2 (dotted, red), and 3 (dashed, blue), cf. Eq. (22), computed for different system size $N = 5, 10, 20, 40, 80$ (bottom to top). Other parameters: $n_D = 0$.

Eq.(7), for $\alpha = 1, 2, 3$ in the corresponding regimes for $\bar{n} \rightarrow 0$ or $\bar{n} \rightarrow \infty$. In the limit of vanishing source occupation $n_S \rightarrow 0$, both $\langle J_z^2 \rangle \approx \langle J_z \rangle^2$ and $\langle J_z^3 \rangle \approx \langle J_z^2 \rangle \langle J_z \rangle$ become exact. In the limit of infinite source occupation $n_S \rightarrow \infty$ however, all odd expectation values $\langle J_z^{2\alpha+1} \rangle$ are vanishing, while even expectation values are scaling with powers of the system size N . Thus, $\langle J_z^2 \rangle \approx \langle J_z \rangle^2$ is not a good approximation in this regime, while $\langle J_z^3 \rangle \approx \langle J_z^2 \rangle \langle J_z \rangle$ again gets exact, explaining the failure of approximation 2.

The validity of the general approximation Eq. (16) can be clarified by calculating

$$\begin{aligned} \langle e^{in\chi} J_z^\alpha \rangle &= \sum_{n=-\infty}^{\infty} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}} \langle n, m | e^{in\chi} J_z^\alpha \rho | n, m \rangle \quad (27) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}} e^{in\chi} \langle m | J_z^\alpha \rho^{(n)} | m \rangle \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}} (2m)^\alpha e^{in\chi} \langle m | \rho^{(n)} | m \rangle, \end{aligned}$$

where $\rho^{(n)} = \langle n | \rho | n \rangle$ is the conditioned density matrix, and in the last line the eigenvalue of J_z^α was inserted. Obviously, the expectation value can be factored when the system is close to the ground state $\langle m | \rho^{(n)} | m \rangle \approx \delta_{m, -\frac{N}{2}} C_n$, which is the case in the limit $\bar{n} \rightarrow 0$ (e.g. low thermobias and $n_D = 0$). Alternatively, factorization is possible when the system is equipartitioned $\langle m | \rho^{(n)} | m \rangle \approx C_n$, which is fulfilled for the limit

$\bar{n} \rightarrow \infty$ (e.g. large thermobias). This explains the good performance of the approximated solutions in the regimes of low and high weighted average occupations \bar{n} .

In contrast, in the intermediate region of \bar{n} neither fluctuations are suppressed due to the system being close to its ground state, nor the supply of bosons from the source is large enough to re-pump the large spin J_z to its thermalized state. Thus, the dynamics is governed by higher order correlations in the intermediate regime that are not taken into account at this level of factorization.

V. CONCLUSION

The application of the equation of motion method to the CGF on top of the master equation allows for the approximate calculation of the full dynamics of the system. For the special case $N = 1$ the method becomes exact and recovers the FCS results obtained by diagonalization of the full Liouvillian. In the limits of low and high thermo-bias the master equation results of previous work [16] are recovered for the full long-term CGF. Furthermore, this method enables us to directly access the thermodynamic limit, which is not generally possible in the master equation solution.

For the approximate solutions we showed that it is nec-

essary to take at least the equation for $\langle J_z^2 \rangle$ into account and factorize at the level of $\langle J_z^3 \rangle$, to get comparable results to the master equation in the stationary limit.

Future work should be directed to the question whether the inclusion of higher correlations of $\langle e^{inx} J_z^\alpha \rangle$ into the equation for the CGF yields better results for higher cumulants and also at intermediate source/drain occupations.

As is evident from the case of $N = 1$, one should in general carefully examine whether for system operators A our main approximation $\langle e^{inx} A \rangle \approx \langle e^{inx} \rangle \langle A \rangle$ is sufficient, or if one has to close the system of equations at higher order. However, since the zeroth order of the method allows for closure with only a few equations, and many relevant probability distributions are close to Gaussian and thus governed by the first two cumulants [27], this approach could find application in a broader community.

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- [1] Y. M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
 - [2] G. Ferrini, A. Minguzzi and F. W. J. Hekking, Phys. Rev. A **80**, 043628 (2009).
 - [3] J. Gabelli and B. Reulet, Phys. Rev. B **80**, 161203 (2009).
 - [4] R. J. Cook, Phys. Rev. A **23**, 1243 (1981).
 - [5] F. Schlögl and E. Schöll, Z. Phys. B **51**, 61 (1983).
 - [6] L. S. Levitov, H. Lee and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).
 - [7] D. A. Bagrets and Y. V. Nazarov, Phys. Rev. B **67**, 085316 (2003).
 - [8] C. Flindt *et al.*, Proc. Natl. Acad. Sci. U.S.A. **106**, 10116 (2009).
 - [9] A. Andreev and A. Kamenev, Phys. Rev. Lett. **85**, 1294 (2000).
 - [10] M. Richter, A. Carmele, A. Sitek and A. Knorr, Phys. Rev. Lett. **103**, 087407 (2009).
 - [11] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
 - [12] M. Gross and S. Haroche, Phys. Rep. **93**, 301 (1982).
 - [13] R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A **4**, 302 (1971).
 - [14] F. Haake and R. J. Glauber, Phys. Rev. A **5**, 1457 (1972).
 - [15] P. A. Braun, D. Braun, F. Haake, and J. Weber, Eur. Phys. J. D **2**, 165 (1998).
 - [16] M. Vogl, G. Schaller, and T. Brandes, Ann. Phys. (N.Y.) **326**, 2827 (2011).
 - [17] T. Brandes, Phys. Rep. **408**, 315 (2005).
 - [18] R. H. Hadfield, Nat. Phot. **3**, 696 (2009).
 - [19] S. De Liberato, N. Lambert, and F. Nori, Phys. Rev. A **83**, 033809 (2011).
 - [20] I. S. Grudinin, H. Lee, O. Painter, K. J. Vahala, Phys. Rev. Lett. **104**, 083901 (2010).
 - [21] J. G. Bohnet *et al.*, Nature **484**, 78 (2012).
 - [22] G. Schaller, G. Kießlich and T. Brandes, Phys. Rev. B **80**, 245107 (2009).
 - [23] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, (Oxford University Press, Oxford, 2002).
 - [24] G. Schaller, Phys. Rev. E **83**, 031111 (2011).
 - [25] G. S. Agarwal, *Quantum Statistical Theories of Spontaneous Emission and their relation to other approaches*, Springer Tracts in Modern Physics Vol. 70 (Springer, Berlin, 1974).
 - [26] C. Flindt, T. Novotny, A. Braggio, M. Sassetti and A.P. Jauho, Phys. Rev. Lett. **100**, 150601 (2008).
 - [27] T. Karzig and F. von Oppen, Phys. Rev. B **81** 045317 (2010).